Title: "ON THE DISTRIBUTION LAWS OF THE 'COMPLETION NUMBERS' IN QUANTUM STATISTICS" USSR

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"ON THE DISTRIBUTION LAWS OF THE '~~FILLING-UP~~ COMPLETION NUMBERS' IN QUANTUM STATISTICS"

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The aim of the present article is to show that ~~the~~ my development ~~by~~ me/in recent years of the method of ~~reasoning~~ deriving the asymptotic formulas of statistical physics with the aid of the limit theorems of the theory of probabilities allows one to establish easily the limit laws of distribution for so-called "completion numbers", for which the usual expositions give only the mean values and dispersion. We shall consider a system of identical particles (the number of particles is N, energy is E, and volume is V) which obey any of the three main statistical schemes: Maxwell-Boltzmann (P), Bose-Einstein (S), and Fermi-Dirac (A). The energy levels of the particles are assumed to be integral numbers; to the level r corresponds Vg_r of the various linearly independent states of a particle, which are designated (in any order) by $u_{r,1}, u_{r,2}, \ldots, u_{r,Vg_r}$. The completion number a_{rs} is the number of particles found in the state u_{rs} ($s = 1,2,\ldots,Vg_r$). See my book entitled Mathematical Foundations of Quantum Statistics, 1951, for the other terminology, designations, and premises.

The total number of linearly independent states of a system for given N,E,V is designated by $\Omega(N,E)$. If m is an integral non-negative number, then the (microcanonical) probability $P(a_{rs} = m)$ of the equality is the ratio of the number $\Omega_{rs}(m)$ of states of the system, in which states we have $a_{rs} = m$, to the number $\Omega(N,E)$ of all states of the system. Our aim is to find the limit of this probability under the condition that N,E,V increase without limit preserving a constant ratio among themselves.

Along with the set u_{kl} ($k = 1,2,\ldots$; $l = 1,2,\ldots,Vg_k$) of states of a particle we shall consider another set obtained from the first set by rejecting the state u_{rs}

(so that to the energy level r corresponds no longer Vg_r , but $Vg_r - 1$ of the linearly independent states); and we shall designate by the symbol * all quantities constructed on the basis of this second set. Then, the elementary combinatorial considerations easily give for the main three statistical schemes the following expressions:

$$P(a_{rs}=m) = \binom{N}{m} \frac{\Omega^*(N-m, E-mr)}{\Omega(N,E)} \qquad (P)$$

$$P(a_{rs}=m) = \frac{\Omega^*(N-m, E-mr)}{\Omega(N,E)} \qquad (S)$$

$$P(a_{rs}=m) = \begin{cases} \dfrac{\Omega^*(N-m, E-mr)}{\Omega(N,E)} & (m \leqslant 1) \\ 0 & (m > 1) \end{cases} . \qquad (A)$$

In the above-mentioned book ~~of mine~~ by myself, Formulas (22) page 191 and (37) page 203 give for any α and $\beta > 0$ and for any integers $p \geqslant 0$, $q \geqslant 0$, for which $\Omega(p,q) \neq 0$, in any of the three statistical schemes the following expression:

$$\Omega(p,q) = C(p)\,\Phi(\alpha,\beta)e^{\alpha p + \beta q}\left(d/2\pi V\delta^{\frac{1}{2}} + O\left(\frac{1+u^2}{V^2}\right) \right) . \qquad (1)$$

Here $C(p) = p!$ (P), $C(p) = 1$ (S,A), and

$$\ln \Phi(\alpha,\beta) = \begin{cases} V\sum_{r=1}^{\infty} g_r \cdot \exp-(\alpha+\beta r) & (P) \\ -V\sum_{r=1}^{\infty} g_r \cdot \ln\left(1 - \exp-(\alpha+\beta r)\right) & (S) \\ V\sum_{r=1}^{\infty} g_r \cdot \ln\left(1 + \exp-(\alpha+\beta r)\right) & (A) \end{cases} \qquad (2)$$

$$u = \max(|u_1|,|u_2|), \quad u_1 = p + \partial \ln \Phi/\partial\alpha, \quad u_2 = q + \partial \ln\Phi/\partial\beta$$

$$\delta = V^{-2}\left((\partial^2 \ln\Phi/\partial\alpha^2)\cdot(\partial^2\ln\Phi/\partial\beta^2) - (\partial^2 \ln\Phi/\partial\alpha\,\partial\beta)^2\right) \qquad (3)$$

and d is a natural number depending only on the structure of the particles. As usual we select α and β so that $N + \partial \ln\Phi/\partial\alpha = 0$, $E = -\partial \ln\Phi/\partial\beta$. Then for $p = N$, $q = E$ we have $u_1 = u_2 = 0$, and Formula (1) gives (for all three statistics) the following expression:

$$\Omega(N,E) = C(N)\,\Phi(\alpha,\beta)\cdot e^{\alpha N + \beta E}\left(d/2\pi V\delta^{\frac{1}{2}} + O(V^{-2}) \right) .$$

In order to evaluate $\Omega^*(N - m, E - mr)$ we must replace Φ and δ respectively by Φ^* and δ^* in Formula (1) and set

$$u_1 = N - m + \partial \ln \Phi^*/\partial\alpha = \partial \ln \Phi^*/\partial\alpha - \partial \ln \Phi\, \partial\alpha - m$$

$$u_2 = E - mr + \partial \ln \Phi^*/\partial\beta = \partial \ln \Phi^*/\partial\beta - \partial \ln \Phi\, \partial\beta - mr$$

but

$$\partial \ln \Phi^*/\partial\alpha - \partial \ln \Phi/\partial\alpha = T_r = 1/(e^{\alpha + \beta r} - \sigma)$$

$$\partial \ln \Phi^*/\partial\beta - \partial \ln \Phi/\partial\beta = rT_r, \text{ where } \sigma = 0(P), \ 1(S), \ -1(A);$$

thus,

$$u_1 = T_r - m, \quad u_2 = r(T_r - m), \quad u = r/T_r - m/.$$

For $N \to \infty$ u remains constant, and we find in any of the three schemes the following:

$$\Omega^*(N-m, E-mr) = C(N-m)\,\Phi^*(\alpha,\beta)\cdot e^{\alpha(N-mr) + \beta(E-mr)}\cdot\left(\frac{d}{2\pi\cdot V\sqrt{\delta}} + O(V^{-2})\right).$$

Thus we obtain for any of the three schemes the following expression:

$$P(a_{rs}=m) = \frac{1}{C(m)}\cdot\frac{\Phi^*(\alpha,\beta)}{\Phi(\alpha,\beta)}\cdot e^{-(\alpha+\beta r)m}\,(1 + O(V^{-1}))$$

(since $\delta/\delta^* = 1 + O(1/V)$, which easily follows from Formula (3)).

In the case of (P) we have $C(m) = m!$, and from Formula (2) it follows that

$$\ln \Phi^*(\alpha,\beta)/\Phi(\alpha,\beta) = -\exp-(\alpha + \beta r) = -T_r,$$

so that

$$P(a_{rs}=m) = e^{-T_r}\cdot(T_r^m/m!)\cdot(1 + O(V^{-1})) \qquad (P)$$

(Poisson's law with parameter T_r).

In the case of (S) we have $C(m) = 1$, and from Formula (2) it follows that:

$$\ln \Phi^*(\alpha,\beta)/\Phi(\alpha,\beta) = \ln(1 - \exp-(\alpha + \beta r)),$$

so that

$$P(a_{rs}=m) = (1 - \exp-(\alpha + \beta r))\cdot e^{-(\alpha + \beta r)m}\cdot(1 + O(V^{-1}))$$
$$= (T_r^m/(T_r+1)^{m+1})\cdot(1 + O(V^{-1})) \qquad (S)$$

(the indicial law).

Finally in the case of (A) we have $C(m) = 1$ and the following:

$$\ln \Phi^*/\Phi = -\ln(1 + \exp-(\alpha + \beta r)) \qquad (\text{where } \Phi \equiv \Phi(\alpha,\beta) \text{ etc})$$

and we easily show that

$$P(a_{rs}= 0) = (1 - T_r)(1 + O(V^{-1})), \quad P(a_{rs}= 1) = T_r(1 + O(V^{-1})); \qquad (A)$$

finally $P(a_{rs}=m) = 0$ for $m > 1$.

The result (A), obviously, is trivial, since, in the case where a_{rs} can assume only the values 0 and 1, $P(a_{rs} = 1)$ equals the mathematical expectation of the number a_{rs} which, as has been known for some time, coincides in the limit with T_r.

It stands to reason that the expressions for the limit values of moments and central moments of the numbers a_{rs} are elementary consequences of the found limits of the distribution laws.

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1. A. Ya. Khinchin, Matematicheskiye Osnovaniya Kvantovoy Statistiki, 1951.

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